Write your name here

Surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Further Mathematics

Advanced

Paper 2: Core Pure Mathematics 2

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference

9FM0/02

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ►

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Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

realising quickly that this is a 'related expressions' question - hence using the roots of polynomial equations formulae on the given cubic

$$\overset{}{x^3} \underset{}{-} \overset{\Sigma\alpha}{8x^2} + \overset{\Sigma\alpha\beta}{28x} - \overset{\alpha\beta\gamma}{32} = 0$$

sum of roots $= \Sigma\alpha = -b/a = -(-8) = 8$

sum of product pairs $= \Sigma\alpha\beta = c/a = 28$

product of roots $= \alpha\beta\gamma = -d/a = -(-32)$
$= 32$

...and use these to solve below:

(i) $\left.\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}\right\}$ sum of reciprocals - get common denominator

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{28^{\div 4}}{32_{\div 4}} = \boxed{\frac{7}{8}}$$

(ii) $(\alpha+2)(\beta+2)(\gamma+2)$ - WAY 1: expand the brackets

$$(\alpha\beta + 2\alpha + 2\beta + 4)(\gamma + 2)$$
$$= (\alpha\beta\gamma + 2\alpha\gamma + 2\beta\gamma + 4\gamma + 2\alpha\beta + 4\alpha + 4\beta + 8)$$

and try get formulea out

$$= \alpha\beta\gamma + 2(\Sigma\alpha\beta) + 4(\Sigma\alpha) + 8$$
$$= 32 + 2(28) + 4(8) + 8$$
$$= 32 + 56 + 32 + 8 \quad = \boxed{128}$$

WAY 2: treat as 'linear transformation of roots

let $w = x+2 \Rightarrow x = w-2$

subbing this into cubic

$$(w-2)^3 - 8(w-2)^2 + 28(w-2) - 32 = 0$$

expand - BINOMIAL

... Pascal's triangle:

1
1 1
1 2 1
1 3 3 1

$$[1w^3 + 3w^2(-2) + 3w(-2)^2 + 1(-2)^3] - 8[w^2 - 4w + 4] + 28w - 56 - 32 = 0$$

$$w^3 - 6w^2 + 12w - 8 - 8w^2 + 32w - 32 + 28w - 56 - 32 = 0$$

collect like terms

$$w^3 - 14w^2 + 72w - 128 = 0$$

↳ this is a transformed cubic of which roots are $(\alpha+2)$, $(\beta+2)$, $(\gamma+2)$ - need to evaluate their product

$$\Rightarrow (\alpha+2)(\beta+2)(\gamma+2) = -(-128)$$

$$\boxed{= 128}$$

(iii) notice sum of squares - so using memorised $\alpha^2+\beta^2+\gamma^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

OR rearranging

$$(\alpha+\beta+\gamma)^2 = (\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$$
$$= \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2$$
$$= \alpha^2 + \beta^2 + \gamma^2 + 2(\Sigma\alpha\beta)$$

... to make $\alpha^2+\beta^2+\gamma^2$ the subject,

$$\alpha^2+\beta^2+\gamma^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$$
$$= (8)^2 - 2(28)$$
$$= 64 - 56$$
$$\boxed{= 8}$$

Question 1 continued

(Total for Question 1 is 8 marks)

2. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1 **(3)**

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2 **(2)**

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree. **(3)**

(a) question is asking us to find the perp. distance from a point to a plane – first need π_1 in Cartesian form $\Rightarrow 3x - 4y + 2z - 5 = 0$

WAY 1: using the formula from formula booklet

perp. distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is

$$d = \frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

$$\Rightarrow d = \frac{|3(6) - 4(2) + 2(12) - 5|}{\sqrt{(3)^2 + (-4)^2 + (2)^2}} = \frac{29}{\sqrt{29}}$$ rationalise $\times\sqrt{29}$ / $\times\sqrt{29}$

$$= \frac{\cancel{29}\sqrt{29}}{\cancel{29}} \boxed{= \sqrt{29}}$$

WAY 2: separating 'shortest distances'

first need 'shortest distance from origin to plane' i.e $\frac{p}{|n|}$ ← the 'p' from scalar product formula of π_1 $(r.n = p)$

$$|n| = \sqrt{(-3)^2 + (-4)^2 + (2)^2}$$

$$= \sqrt{9 + 16 + 4} = \sqrt{29}$$

$$\therefore \frac{5}{\sqrt{29}}$$

next need perp. distance from plane containing $\begin{pmatrix}6\\2\\12\end{pmatrix}$ to origin

i.e $\frac{a \cdot n}{|n|}$

$$= \frac{\begin{pmatrix}6\\2\\12\end{pmatrix} \cdot \begin{pmatrix}3\\-4\\2\end{pmatrix}}{\sqrt{29}} = \frac{6(3) + 2(-4) + 12(2)}{\sqrt{29}}$$ expand numerator $\frac{18 - 8 + 24}{\sqrt{29}}$

Question 2 continued

$$= \frac{34}{\sqrt{29}}$$

$$\therefore \text{perp. distance} = \left| \frac{a \cdot n}{\sqrt{n}} - \frac{p}{|n|} \right|$$

$$= \frac{34}{\sqrt{29}} - \frac{5}{\sqrt{29}} = \frac{29}{\sqrt{29}} \begin{matrix} \times\sqrt{29} \\ \times\sqrt{29} \end{matrix}$$

$$= \frac{\cancel{29}\sqrt{29}}{\cancel{29}} \boxed{= \sqrt{29}}$$

(b) if $\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ is perpendicular to Π_2 then the dot product between $\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ and the two direction vectors would have to be equal to 0

$$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = -1(2) + 1(-3) + 1(5) = 0$$

$$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -1(1) + (-3)(-1) + 1(-2) = 0$$

$$\boxed{\therefore \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \text{ is perpendicular to } \Pi_2}$$

(c) using formula for acute angles between 2 planes : $\cos\theta = \left| \frac{n_1 \cdot n_2}{\sqrt{n_1^2} + \sqrt{n_2^2}} \right|$

& using fact that $\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ is perpendicular -from (b) - it must be the normal

$$\cos\theta = \left| \frac{\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + (-4)^2 + (2)^2}} \right|$$

$$\Rightarrow \cos\theta = \frac{3(-1) + (-4)(-3) + 2(1)}{\sqrt{29}\sqrt{11}} = \frac{11}{\sqrt{29 \times 11}} = \frac{11}{\sqrt{319}} \begin{matrix} \times\sqrt{319} \\ \times\sqrt{319} \end{matrix}$$

$$= \frac{\cancel{11}\sqrt{319}}{\cancel{319}_{29}} = \frac{\sqrt{319}}{29}$$

taking $\cos^{-1}$ of each side

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{319}}{29}\right)$$

$$= 51.98398\ldots$$

$$\boxed{= 52^\circ \text{ (2 s.f)}}$$

(Total for Question 2 is 8 marks)

DO NOT WRITE IN THIS AREA

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

(i)(a) remembering that for M to have an inverse it has to be non-singular ($\det(M) \neq 0$)

$$\det(M) = 2\begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} - a\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 2(3) - a(-2) + 4(1)$$

$$= 6 + 2a + 4$$

$$= 2a + 10 \neq 0$$

$$\Rightarrow 2a \neq -10$$

$\div 2$ $\quad$ $\div 2$

$$\boxed{a \neq 5}$$

(b) step 1: find $\det(M)$

$= 2a + 10$

step 2: find the matrix of MINORS - call it A - matrix where elements are replaced by the determinant of the 2×2 matrix left after all elements corresponding to the elements are removed

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & 4+a \\ 4-a & -6 & -2-a \end{pmatrix}$$

step 3: find the matrix of COFACTORS - call it C - change the sign of the elements with the -ve

$$\begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{pmatrix} \Rightarrow C = \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -4-a \\ 4-a & 6 & -2-a \end{pmatrix}$$

Step 4: C^T i.e keep the main diagonal and switch the positions of the highlighted

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{i.e} \quad \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -4-a \\ 4-a & 6 & -2-a \end{pmatrix}$$

$$\Rightarrow C^T = \begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -4-a & -2-a \end{pmatrix}$$

Step 5: $M^{-1} = \frac{1}{\det(M)} C^T$

$$M^{-1} = \boxed{\frac{1}{2a+10} \begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -4-a & -2-a \end{pmatrix}}$$

(ii) to prove by induction means have to prove a conjecture is true for all $n \in \mathbb{N}$

Step 1: base case : prove true for $n=1$

LHS: $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ RHS: $\begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

LHS = RHS $\therefore$ true for $n=1$

Step 2: assumption case : assume true for $n=k$

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$$

Step 3: induction step : prove true for $n=k+1$

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

AIM:

$$\begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$$

Sub in assumption step

$$= \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

matrix multiplication : multiply the elements in the row of the first matrix by the elements in the column of the second matrix and sum in between

Repeat (see colours to indicate)

Question 3 continued

$$\begin{pmatrix} \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} & \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \\ \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} & \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} 3^k(3)+0(6) & 3^k(0)+0(1) \\ 3(3^k-1)(3)+1(6) & 3(3^k-1)(0)+(1)(1) \end{pmatrix}$$

↳ this requires further manipulation to equate to the aim

using index laws: $a^m \times a^n = a^{m+n}$

$\Rightarrow 3^k(3) = 3^{k+1}$

multiply inside of bracket by one of the threes and factorise the other 3 out

$= 3(3^1 \times 3^k - 1\times 3 + 2)$

use index laws : $a^m \times a^n = a^{m+n}$

$= 3(3^{k+1} - 3 + 2) = 3(3^{k+1} - 1)$

$$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix} = \text{AIM}(\checkmark)$$

$\therefore$ true for $n=1$

Step 4: conclusion step: since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $k \in \mathbb{N}$

(Total for Question 3 is 12 marks)

4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

(a) evaluate LHS using mod-arg form : $z = \cos\theta + i\sin\theta$

...using DMT :

z^n (using DMT) $= \cos n\theta + i\sin n\theta$

AND $z^{-n} = \cos(-n)\theta + i\sin(-n)\theta$

↳ using cos - even function ⇒ $\cos(-\theta) = \cos\theta$

sin - odd function ⇒ $\sin(-\theta) = -\sin\theta$

⇒ $z^{-n} = \cos n\theta - \sin n\theta$

subbing into LHS

$z^n + \frac{1}{z^n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$

$\boxed{= 2\cos n\theta = RHS}$

(b) now question is asking us to express high trig powers into multi-angle form - first rewriting LHS using standard $z^1 + z^{-1}$ formulae raised to the power of 4

$(z + z^{-1})^4 = (2\cos\theta)^4$

$= 16\cos^4\theta$

and eval. LHS using BINOMIAL EXPANSION

... Pascal's: $1z^4 + 4z^3(z^{-1}) + 6z^2(z^{-2}) + 4z(z^{-3}) + 1(z^{-4})$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

evaluate index laws

$z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

factorise the 'common powers'

$16\cos^4\theta = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

and using $z^n + z^{-n} = 2\cos n\theta$ from part (a)

⇒ $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\div 2$ $\quad 8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$ $\div 2$

Question 4 continued

$\div 8$ $\qquad$ $\div 8$

$$\boxed{\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)}$$

(Total for Question 4 is 7 marks)

5.

$$y = \sin x \sinh x$$

(a) Show that $\dfrac{d^4y}{dx^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

(a) we're asked to evaluate a higher derivative of a product of two expressions (including a hyperbolic) - know we're going to have to use product rule

$y = \sin x \sinh x$

differentiate using product rule - use $\frac{d}{dx}\sinh x = \cosh x$

$\frac{dy}{dx} = \cos x \sinh x + \sin x \cosh x$

differentiate again - product rule - use $\frac{d}{dx}\cosh x = \sinh x$

$\frac{d^2y}{dx^2} = -\cancel{\sin x \sinh x} + \cos x \cosh x + \cancel{\sin x \sinh x} + \cos x \cosh x$

$= 2\cos x \cosh x$

differentiate again - product rule

$\frac{d^3y}{dx^3} = 2\cos x \sinh x - 2\sin x \cosh x$

differentiate a final time

$\frac{d^4y}{dx^4} = \cancel{2\cos x \cosh x} - 2\sin x \sinh x - 2\sin x \sinh x - \cancel{2\cos x \cosh x}$

$= -4\sin x \sinh x$

but answer is in terms of 'y', so have to LINK 4th derivative back to 'y'

$\boxed{\Rightarrow \frac{d^4y}{dx^4} = -4y}$

(b) recognising the Maclaurin series for a function as an infinitely long polynomial where the COEFFICIENTS of the power of 'x' and its derivatives are all evaluated as zero

...from formula booklet : $y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{x^r}{r!}f^{(r)}(0)$

evaluate higher derivatives from part (a) at 0

let $f(x) = \sin x \sinh x$ $\quad f(0) = 0$

$f'(x) = \sin x \cosh x + \sinh x \cos x$ $\quad f'(0) = 0$

$f''(x) = 2\cos x \cosh x$ $\quad f''(0) = 2$

$f'''(x) = 2\cos x \sinh x - 2\sin x \cosh x$ $\quad f'''(0) = 0$

$f^4(x) = -4f(x)$ $\quad f^4(0) = 0$

because we need first three non-zero terms - keep differentiating the 4th derivative to avoid using long notation all the time

$f^{(5)}(x) = -4f'(x)$ $\quad f^{(5)}(0) = -4(0) = 0$

$f^{(6)}(x) = -4f''(x)$ $\quad f^{(6)}(0) = -4(2) = -8$

WAY 1: long way - rest of derivatives

$f^{(7)}(x) = -4f'''(x)$ $\quad f^{(7)}(0) = -4(0) = 0$

$f^{(8)}(x) = -4f^{(4)}(x)$ $\quad f^{(8)}(0) = 0$

$f^{(9)}(x) = -4f^{(5)}(x)$ $\quad f^{(9)}(0) = 0$

$f^{(10)}(x) = -4f^{(6)}(x)$ $\quad f^{(10)}(0) = 32$

WAY 2: notice PATTERN

see second derivative $= 2$, then sixth derivative $= -4(2)$
$= -8$

∴ following arithmetic sequence in derivatives ($d = 4$) and geometric in term coefficients ($r = -4$)

it'd have to be tenth derivative

$\times -4 \, (-8)$

$\Rightarrow f^{(10)}(x) = 32$

Subbing into formula book Maclaurin series

$$f(x) = \frac{2}{2!}x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10}$$

$$\Rightarrow \boxed{f(x) = x^2 - \frac{x^6}{90} + \frac{x^{10}}{113{,}400}}$$

(c) notice that $\frac{d^2y}{dx^2} = 2$

$\frac{d^4y}{dx^4} = -8$

$\frac{d^{10}y}{dx^{10}} = 32$

for 'nth non zero term' - check powers

$(-2) \underset{-4}{\frown} 2 \underset{+4}{\frown} 6 \underset{+4}{\frown} 10$ } arithmetic sequence - nth term is $4n - 2$

so far, have $\frac{x^{4n-2}}{(4n-2)!}$

now - check coefficients

$2 \xrightarrow{\times -4} -8 \xrightarrow{\times -4} 32$ } forms a geometric sequence - nth term from $u_n = ar^{n-1} \Rightarrow 2(-4)^{n-1}$

Question 5 continued

hence combining the two for first three non-zero terms :

$$\boxed{= 2(-4)^{n-1} \frac{x^{4n-2}}{(4n-2)!}}$$

(Total for Question 5 is 10 marks)

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathrm{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \{z : |z - 4 - 3\mathrm{i}| \leqslant 5\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

(a)(i) because the given locus is in the form $|z-(a+bi)|=r$ which usually represents circle centre (a, b), radius r

$\therefore$ given locus must represent circle centre $(4, 3)$, radius 5 –

to sketch this in exam, see if it passes through the origin

... using Cartesian equtn for circle (general $(x-a)^2+(y-b)^2=r^2$

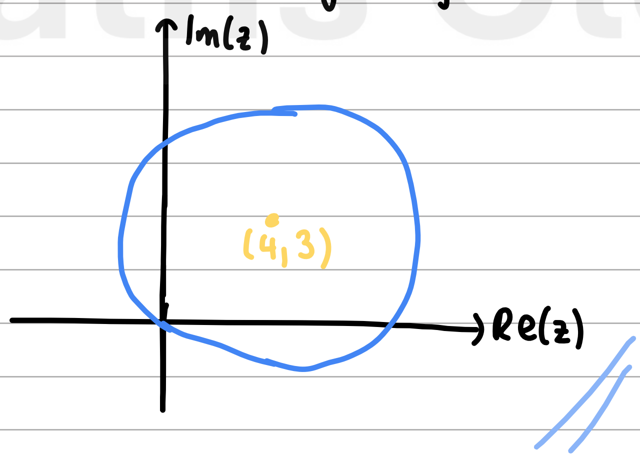
$$(x-4)^2+(y-3)^2=25$$

sub in $(0,0)$

$$(0-4)^2+(0-3)^2=25$$

$$16+9=25 \checkmark$$

$\therefore$ through origin



(b) now the question is basically asking us to convert the Cartesian equation of the circle into POLAR FORM – using cartesian equtn from (a):

$(x-4)^2+(y-3)^2=25$

Sub in POLAR DFN of $(x,y)\to(r\cos\theta, r\sin\theta)$

$$(r\cos\theta-4)^2+(r\sin\theta-3)^2=25$$

expand brackets

$$r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta-6r\sin\theta+9=25$$

collect similar terms

$$r^2\underbrace{(\cos^2\theta+\sin^2\theta)}-8r\cos\theta-6r\sin\theta=0$$

using identity: $\sin^2\theta+\cos^2\theta\equiv1$

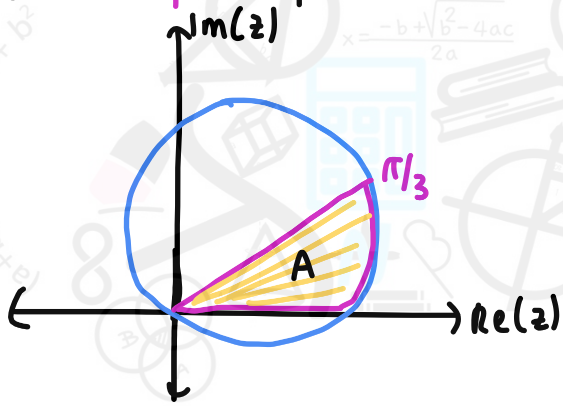
$$r^2=8r\cos\theta+6r\sin\theta$$

$\div r$ $\qquad \div r$

$$\boxed{\Rightarrow r=8\cos\theta+6\sin\theta}$$

(b)(i) recognising that the first loci is in the form $\arg(z-(\alpha+\beta i))=\theta$ which represents a half line that extends from BUT NOT INCLUDING the point (α,β) and making angle θ with a line parallel to the real axis

where a half line from the ORIGIN - between 0 and $\pi/3$



(ii) WAY 1: using polar area integration:

to find A we could integrate the polar curve version of the circle from $(\pi/3, 0)$ - using formula for polar integration.

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/3}(8\cos\theta+6\sin\theta)^2\,d\theta$$

expand inside bracket

$$=\frac{1}{2}\int_0^{\pi/3}(64\cos^2\theta+96\sin\theta\cos\theta+36\sin^2\theta)\,d\theta$$

but know can't really integrate trig powers - rewriting using cos double angle - $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos2\theta$

$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos2\theta$

$$=\frac{1}{2}\int_0^{\pi/3}\left(64\left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)+96\sin\theta\cos\theta+36\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)\right)d\theta$$

using sin double angle

$$= \frac{1}{2}\int_0^{\pi/3}\left(32+32\cos 2\theta+48(2\sin\theta\cos\theta)+18-18\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\left(50+14\cos 2\theta+48\sin 2\theta\right)d\theta$$

integrate using $\int \cos k\theta \, d\theta = \frac{1}{k}\sin k\theta$; $\int \sin k\theta = -\frac{1}{k}\cos k\theta$

$$= \frac{1}{2}\left[50\theta+7\sin 2\theta-24\cos 2\theta\right]_0^{\pi/3}$$

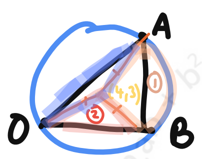
$$= \frac{1}{2}\left\{\left[\frac{50\pi}{3}+7\sin\left(\frac{2\pi}{3}\right)-24\cos\left(\frac{2\pi}{3}\right)\right]-\left[50(0)+7\sin(2\times 0)-24\cos(2\times 0)\right]\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{50\pi}{3}+\frac{7\sqrt{3}}{2}+12\right)-(-24)\right\}$$

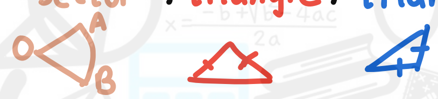
$$= \frac{1}{2}\left(\frac{50\pi}{3}+\frac{7\sqrt{3}}{2}+36\right)$$

$$\boxed{= \frac{25\pi}{3}+\frac{7\sqrt{3}}{4}+18}$$

WAY 2: GEOMETRIC approach



eg. A = sector ① + triangle ② + triangle ③



- first, sector

angle ACB $= \frac{2\pi}{3}$ $\Rightarrow$ area of sector $= \frac{1}{2}(5^2)\left(\frac{2\pi}{3}\right) = \boxed{\frac{25\pi}{3}}$

- next, triangle: first need B where circle intersects Re(z) axis ($y=0$)

$$(x-4)^2+(0-3)^2=25$$

expand brackets

$$x^2-8x+16+9=25$$

$$\Rightarrow x^2-8x=0$$

factorise 'x' out

$$x(x-8)=0$$

$$\Rightarrow x=0 \text{ or } x=8$$

$\therefore$ area of triangle $= \frac{1}{2}(8)(3) = \boxed{12}$

- finally, area of triangle OAC - need $\angle OCA$ - cos rule

$$2\pi-\frac{2\pi}{3}-\cos^{-1}\left(\frac{5^2+5^2-8^2}{2\times 5\times 5}\right)$$

$$\Rightarrow \text{area OAC} = \frac{1}{2}(5)^2\sin\left(\frac{4\pi}{3}-\cos^{-1}\left(-\frac{7}{25}\right)\right)$$

$$= \frac{25}{2}\sin\left(\frac{4\pi}{3}-\cos^{-1}\left(-\frac{7}{25}\right)\right)$$

Question 6 continued

$$= \frac{25}{2}\left(\sin\left(\frac{4\pi}{3}\right)\cos\left(\cos^{-1}\left(-\frac{7}{25}\right)\right) - \cos\frac{4\pi}{3}\sin\left(\cos^{-1}\left(-\frac{7}{25}\right)\right)\right)$$

$$= \frac{25}{2}\left(\left(\frac{7\sqrt{3}}{50} + \frac{1}{2}\sqrt{1-\left(\frac{7}{25}\right)^2}\right)\right) = \boxed{\frac{7\sqrt{3}}{4} + 6}$$

$$A = \frac{25\pi}{3} + 12 + \frac{7\sqrt{3}}{4} + 6$$

$$\boxed{= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18}$$

(Total for Question 6 is 13 marks)

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r \quad - ①$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r \quad - ②$$

(a) Show that $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\dfrac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

(a) see how the 'show that' is in terms of foxes, f - want to eliminate rabbits, r

rearrange ① for 'rabbits'

$0.1r = \frac{df}{dt} - 0.2f$ $\times 10$ $\times 10$

$$* \boxed{r = 10\frac{df}{dt} - 2f}$$

differentiate

$$\boxed{\frac{dr}{dt} = 10\frac{df^2}{dt^2} - 2}$$

and sub into ②

$$10\frac{df^2}{dt^2} - 2 = -0.2f + 0.4\left(10\frac{df}{dt} - 2f\right)$$

$$\Rightarrow 10\frac{df^2}{dt^2} - 2 = -0.2f + 4\frac{df}{dt} - 0.8f$$

Question 7 continued

collect like 'f' and its derivatives and take to LHS

$$10\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$$

$\div 10$ $\div 10$

$$\boxed{\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0}$$

(b) asking us to solve 2ODE in part (a)

$\boxed{A.E}$ $m^2 - 0.6m + 1 = 0$

solve using calc equation solver/ quadratic formula

$$m = \frac{0.6 \pm \sqrt{(0.6)^2 - 4(0.1)}}{2}$$

$\Rightarrow$ $m = 0.3 \pm 0.1i$

↳ two complex roots means have to use the corresponding general solution: $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$\boxed{G.S}$ $f = e^{0.3t}(A\cos 0.1t + B\sin 0.1t)$

(c) differentiate f from part (b) – product rule

$$\frac{df}{dt} = 0.3e^{0.3t}(A\cos 0.1t + B\sin 0.1t) + 0.1e^{0.3t}(B\cos 0.1t - A\sin 0.1t)$$

sub into ① REARRAGED (*)

$$r = 10\left(0.3e^{0.3t}(A\cos 0.1t + B\sin 0.1t) + 0.1e^{0.3t}(B\cos 0.1t - A\sin 0.1t)\right)$$

$$\Rightarrow r = e^{0.3t}\left((3A+B)\cos 0.1t + (3B-A)\sin 0.1t\right) - 2e^{0.3t}(A\cos 0.1t + B\sin 0.1t)$$

$\boxed{G.S}$ $r = e^{0.3t}\left((A+B)\cos 0.1t + (B-A)\sin 0.1t\right)$

(d)(i) subbing initial conditions

at $t=0$, $f = 6$

$$6 = Ae^0\cos 0 + Be^0\sin 0$$

$\Rightarrow \boxed{A = 6}$

Question 7 continued

at $t=0$, $r=20$

$$e^0\left[(A+B)\cos 0+(B-A)\sin 0\right]=20$$

$$\Rightarrow 6+B=20$$

$$\Rightarrow \boxed{B=14}$$

$\boxed{\text{P.I}}$

$$f=e^{0.3t}(6\cos 0.1t+14\sin 0.1t)$$

$$r=e^{0.3t}(20\cos 0.1t+8\sin 0.1t)$$

but the year in which rabbits die out requires us to find the 't' at which $r=0$

$$e^{0.3t}(20\cos 0.1t+8\sin 0.1t)=0$$

making each bracket equal 0

$e^{0.3t}\neq 0$

due to exponential properties

$$20\cos 0.1t+8\sin 0.1t=0$$

$$20\cos 0.1t=-8\sin 0.1t$$

$\div\cos 0.1t$

$$20=\frac{-8\sin 0.1t}{\cos 0.1t}$$

$\div -8$ $\quad \div -8$

$$\tan 0.1t=-\frac{20}{8}$$

$$0.1t=\tan^{-1}\left(-\frac{20}{8}\right)$$

$$=-1.19\ldots,\ 1.95\ldots$$

$\div 0.1$ $\quad \div 0.1$

$$\Rightarrow 19.5\ldots\ (t>0)$$

yrs

$\therefore$ the year is $2000+19.5\ldots$

$$\Rightarrow \boxed{2019}$$

(ii) evaluate foxes P.S when $t=19.5\ldots$

$$f=e^{0.3(19.5\ldots)}(6\cos(0.1\times 19.5\ldots)+14\sin(0.1\times 19.5\ldots))$$

$$\Rightarrow f=3754.47\ldots$$

$$\boxed{=3750}$$

(iii) model predicts large no. of foxes on island - but once rabbits (its PREY) die out the model may not be suitable

Question 7 continued

(Total for Question 7 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS